Stability of polytopic systems via convex $M$-matrices and parameter-dependent Lyapunov functions*

D.M. Stipanović, D.D. Šiljak

Department of Electrical Engineering, Santa Clara University, Santa Clara, CA 95051, USA

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1. Introduction

It has been long recognized by Fan [9] that the sum of two matrices is not necessarily an $M$-matrix. In the same paper, Fan proved that the proportional domination of one matrix over the other is a sufficient condition for the $M$-matrix property of their sum to hold. This turned out to be all one needed to establish the $M$-matrix property for a convex combination of a set of matrices and, thus, show connective stability of an uncertain matrix with respect to a convex polytope in the coefficient space [30]. A natural way to strengthen this result was to use the necessary and sufficient condition of Horn and Johnson [16] for a convex combination of a pair of $M$-matrices to be an $M$-matrix. However, a generalization of the condition to a polytope of matrices remains an open problem.

The main objective of this paper is to derive a number of sufficient conditions for the convex combination of a set of $M$-matrices to be an $M$-matrix. The conditions are obtained using as diverse tools as $M$-matrix properties and linear programming, mathematical induction and linear matrix inequalities, and, exceptly, the conditions are independent of each other. They can be used to verify with a varying degree of

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*Corresponding author. Tel.: +1-408-554-4469.
E-mail address: ddsiljak@scu.edu (D.D. Šiljak).

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efficiency if all matrices in a given polytope are $M$-matrices by testing only the vertices of the polytope.

A strong motivation for deriving convexity conditions for $M$-matrices is the fact that this class of matrices appears in a wide variety of models for natural and man-made systems. General equilibrium in mathematical economics [2,9]; population models in biology [13,15,23,33], electrical networks [29], chemical and compartmental systems [20], large-scale systems [22,24,33], decentralized control [34], and artificial neural networks [11,26,35] are all models where $M$-matrix properties are used to establish stability. The usefulness of these models in real-world situations is greatly enhanced by requiring that they be robustly stable with respect to parametric (structured) perturbations, which are inevitably present due to modeling uncertainties and external fluctuations within the environment of the models. The concept of connective stability [33] captures the effect of uncertain coefficients of the system matrix on stability of the equilibrium states, which may be fixed or moving under perturbations. By restricting the sum of variations (possibly nonlinear and time varying) of the coefficients to a hyper-rectangle in the coefficient space, one determines connective stability of the equilibrium by showing stability of a constant matrix at a single vertex of the hyperrectangle via $M$-matrix properties.

The purpose of the second part of this paper is to broaden the connective stability concept to include polytopic uncertainty sets using the obtained convex $M$-matrix conditions. System matrix is expressed as a convex combination of the fixed vertex matrices and stability of the polytopic system is established by testing matrices at each vertex of the matrix polytope. We apply the same approach to system matrices that are affine functions of uncertain parameters which are restricted to a convex polytope in the parameter space.

The organization of the paper is as follows. In the next section, we introduce the notation and some basic results concerning $M$-matrices, some of them being new connections of the known results. In Section 3, we derive several sufficient conditions for convexity of $M$-matrices and discuss their implications. These conditions are generalized in Section 4 to a polytope of matrices involved in stability analysis of a constant linear system. Section 5 is devoted to stability analysis of the Lotka-Volterra population model, where the system matrix depends on uncertain parameters. An interesting feature in this context is the use of parameter-dependent Liapunov functions which need to be shown to have desired properties for each fixed parameter vector within the prescribed polytope. In Section 6, we consider stability of systems with matrices that depend on both state and uncertain parameters. Finally, in Section 7, polytopic connective stability of interconnected systems is analyzed via convex $M$-matrices and a parameter-dependent formulation of vector Liapunov functions.

2. Notations and preliminaries

We consider vectors $x=(x_1,x_2,\ldots,x_n)^T$ in $\mathbb{R}^n$ and matrices $A=(a_{ij})$ in $\mathbb{R}^{n\times n}$, where $T$ denotes transpose. When $x$ is nonnegative we write $x \in \mathbb{R}_n^+$. Sets are denoted by
capital bold letters, and the set of positive vectors \( x \) is defined as

\[
R_n^+ = \{ x \in R_n^+ : x_i > 0 \quad \forall i \in [n] \},
\]

where \( n = \{1, 2, \ldots, n\} \).

Our central object is the set \( M_n \) of \( M \)-matrices, which is a subset of the set \( Z_n \) of matrices with nonpositive off-diagonal elements,

\[
Z_n = \{ A \in R^{n \times n} : a_{ij} \leq 0, \quad i \neq j, j \in [n] \}.
\]

Before we provide a number of characterizations of \( M \)-matrices that we use in this paper, we recall the standard componentwise order. For two matrices \( A, B \in R^{n \times n} \), \( A \geq B \) means \( a_{ij} \geq b_{ij} \), and \( A > B \) implies \( a_{ij} > b_{ij} \) for all \( i, j \in [n] \).

Then a long list of \( M \)-matrix properties [6,10] we need a few:

**Theorem 2.1.** Let \( A \in Z_n \). Then, the following statements are equivalent:

(i) \( A \in M_n \).

(ii) There is a vector \( x \in R_n^+ \) such that \( Ax \in R_n^+ \).

(iii) There is a vector \( x \in R_n^+ \) such that \( A^T x \in R_n^+ \).

(iv) There is a vector \( x \in R_n^+ \) such that \( A^T x \in R_n^+ \).

(v) There is a positive diagonal matrix \( D = \text{diag}(d_1, d_2, \ldots, d_n) \) such that the matrix

\[
C = DAD
\]

is positive definite. Matrix \( D \) is called the positive diagonal Laplace solution.

(vi) \( A \) is nonsingular and \( A^{-1} \geq 0 \).

(vii) All eigenvalues of \( A \) have positive real parts, that is, \( A \) is positive stable.

To show how (ii)-(v) are related to (vi), we assume that \( A \in M_n \) and define two sets of vectors in \( R^n \):

\[
S(A) = \{ x^T x : x \in R_n^+ \} \quad \text{and} \quad \tilde{S}(A) = \{ (A^T)^{-1} x^T : x \in R_n^+ \}
\]

and prove the following:

**Theorem 2.2.** Let \( A \in M_n \). Then, \( d \in R_n^+ \) with \( Ad \in R_n^+ \) if and only if \( d \in S(A) \).

**Proof.** Observe that \( d \in S(A) \) implies that there exists a vector \( c \in R_n^+ \) such that \( d = A^{-1} c \). Then, \( Ad = AA^{-1} c = c \in R_n^+ \). Conversely, if \( d \in R_n^+ \), then we can set \( c = Ad \in R_n^+ \), which implies \( d \in S(A) \).

Similarly we get:

**Theorem 2.3.** Let \( A \in M_n \). Then, \( \tilde{d} \in R_n^+ \) with \( A^T \tilde{d} \in R_n^+ \) if and only if \( \tilde{d} \in \tilde{S}(A) \).
Theorem 2.4. Let \( A \in \mathbb{M}_n \). If \( d, \tilde{d} \in \mathbb{R}^n \) are such that \( Ad, A^T \tilde{d} \in \mathbb{R}^n \), then \( D = \text{diag}(\tilde{d}, \tilde{d}, \ldots, \tilde{d}) = \text{diag}(d_1, d_2, \ldots, d_n) \) is a positive Liapunov solution.

Proof. Let \( \tilde{D} = \text{diag}(\tilde{d}, \tilde{d}, \ldots, \tilde{d}) \) and \( D = \text{diag}(d_1, d_2, \ldots, d_n) \). Then, \( D = \tilde{D} \tilde{D}^{-1} \) and we obtain
\[
C = A^T \tilde{D} + \tilde{D} A = \tilde{D}^{-1} (A^T + A) \tilde{D} = \tilde{D}^{-1} \tilde{D} \tilde{D}^{-1} = \tilde{D}^{-1} \tilde{D} \tilde{D}^{-1} = \tilde{D}^{-1} \tilde{D}^{-1} \tilde{D} \tilde{D}^{-1}.
\] (2.5)

We observe that the matrix \( \tilde{W} = \tilde{D} \tilde{A} \tilde{D} = (\tilde{d}_i \tilde{a}_{ij}) \) is both rowwise and columnwise diagonally dominant matrix, because for all \( i \in \mathbb{N} \) we have
\[
w_i = \sum_{j=1}^{n} |w_{ij}| = \sum_{j=1}^{n} w_{ij} = \tilde{d}_i \left( \sum_{j=1}^{n} a_{ij} \tilde{d}_j \right) > 0,
\] (2.6)
\[
w_{ij} = \sum_{j=1}^{n} |w_{ij}| = \sum_{j=1}^{n} w_{ij} = \tilde{d}_j \left( \sum_{i=1}^{n} a_{ij} \tilde{d}_i \right) > 0.
\]

This fact implies that \( \tilde{W}^T + \tilde{W} \) is rowwise or columnwise diagonally dominant and, since \( \tilde{W} \in \mathbb{Z}^n \), we conclude that \( \tilde{W}^T + \tilde{W} \in \mathbb{M}_n \). Furthermore, since \( \tilde{W}^T + \tilde{W} \) is symmetric, it is also positive definite. Then, from (2.5) and definition of \( \tilde{W} = (\tilde{d}_i \tilde{a}_{ij}) \), it follows that the matrix \( C \) is also positive definite. \( \Box \)

Example 2.5. The converse of Theorem 2.4 does not hold in general, as testified by the matrix \( [7] \)
\[
A = \begin{bmatrix} 0.4 & -0.1 \\ -0.7 & 0.5 \end{bmatrix}
\] (2.7)
and its positive Liapunov solution \( D = \text{diag}(0.9, 0.2) \). There are no positive vectors \( d, \tilde{d} \) with \( Ad, A^T \tilde{d} \) positive such that \( D = \tilde{D} \).

When a matrix \( A = (a_{ij}) \) is given, the comparison matrix \( Q = (q_{ij}) \) is uniquely defined as
\[
q_{ij} = \begin{cases} \{ a_{ij} \}, & i = j, \\ \{-a_{ij} \}, & i \neq j. \end{cases}
\] (2.8)

and we recall (e.g., [10]) the following:

Definition 2.6. A matrix \( A \in \mathbb{R}^{n \times n} \) is said to be an \( H \)-matrix if its comparison matrix \( Q = (q_{ij}) \in \mathbb{R}^{n \times n} \) is an \( M \)-matrix.

The class of \( H \)-matrices,
\[
H_n = \{ A \in \mathbb{R}^{n \times n} : Q(A) \in \mathbb{M}_n \}
\] (2.9)
is interesting because all our convexity theorems for \( M \)-matrices carry over to \( H \)-matrices without requiring the stringent structure of the class \( M_n \).
Since our interest is in stability, we recall this known result (e.g., [16]):

**Theorem 2.7.** Let \( A \in \mathbb{H}_n \). Then, \( A \) is positive stable if and only if it has positive diagonal elements, that is, \( a_{ii} > 0 \) for all \( i \in \mathbb{N} \).

The positivity requirement is not overly restrictive because it is a necessary condition for a matrix \( A \in \mathbb{H}_n \) to be positive stable. We note, however, that for a matrix \( A \in \mathbb{H}^{++} \) with positive diagonals, the condition that \( 
abla(A) \in \mathbb{H}_n \) is only sufficient for positive stability, which is not the case when \( A \in \mathbb{H}_n \).

### 3. Convex \( M \)-matrices

It is well known that the sum of two \( M \)-matrices is not necessarily an \( M \)-matrix. An obvious but important implication of this fact is that the convex combination

\[
M(\alpha) = \alpha A + (1 - \alpha) B, \quad \alpha \in [0, 1]
\]

of two \( M \)-matrices \( A \) and \( B \) is generally not an \( M \)-matrix. Additional conditions are required for \( M(\alpha) \) to be an \( M \)-matrix for all \( \alpha \in [0, 1] \).

The first characterization of convex \( M \)-matrices was presented by Fan [9] in terms of proportional dominance.

**Definition 3.1.** Let \( A, B \in \mathbb{R}^{n \times n} \) be two \( M \)-matrices. We say that \( A \) proportionally dominates \( B \) rowwise if

\[
\frac{a_{ij}}{a_{ii}} \geq \frac{b_{ij}}{b_{ii}} \quad \forall i, j \in \mathbb{N}.
\]

Columnwise proportional dominance is defined similarly. The significance of proportional dominance is a result of the following theorem obtained by Fan [9]:

**Theorem 3.2.** If \( A \) proportionally dominates \( B \) rowwise or columnwise, then for any numbers \( \alpha, \beta \geq 0 \) with \( \alpha + \beta = 1 \), the matrix \( M(\alpha, \beta) = \alpha A + \beta B \) is an \( M \)-matrix.

This theorem was used in [35] to establish convexity stability of linear systems inside a convex polytope situated in the space of coefficients of the system matrix. This result opened up a possibility to generalize convexity stability of linear systems beyond the interval characterization of uncertainty structures, as we show in Section 4.

To improve on Fan's result, let us introduce:

**Definition 3.3.** Let \( A, B \in \mathbb{R}^{n \times n} \) be two \( M \)-matrices. We say that \( A \) and \( B \) have a common \( d \) if there exist a \( d \in \mathbb{R}_n^* \) such that \( Ad, Bd \in \mathbb{R}_n^* \).

It is obvious that if \( A \) proportionally dominates \( B \) rowwise (or columnwise), then they have a common \( d \); in fact, \( S(A) \subseteq S(B) \). Similarly, rowwise proportional dominance is a sufficient condition for two \( M \)-matrices to have a common \( d \). Our interest in these facts is motivated by:
Theorem 3.4. If two matrices \( A \) and \( B \) have a common \( d \), then \( M(z) = z + |1 - z|B \) is an \( M \)-matrix for all \( z \in [0, 1] \).

Proof. Observe that \( M(z) \in Z_+ \) for all \( z \in [0, 1] \) and that
\[
M(zA) - zA + (1 - z)B \in R^*_+.
\]
(3.3)

Then, by (ii) of Theorem 2.1, \( M(z) \) is an \( M \)-matrix for all \( z \in [0, 1] \). \( \square \)

Remark 3.5. In a similar way, we can show that if two \( M \)-matrices have a common \( d \), then their convex combination is also an \( M \)-matrix.

To complete these facts regarding \( d \) and \( d \), it is of interest to note that two \( M \)-matrices \( A, B \in R^{n \times n} \) have a common \( d \) if and only if \( S(A) \cap S(B) \neq \emptyset \). Now, \( S(A) \cap S(B) \neq \emptyset \) if and only if there is a vector \( c \in R^*_+ \) such that \( A^t c < e \) and \( B^t c < e \). Similarly, two \( M \)-matrices \( A, B \in R^{n \times n} \) have the same \( \delta \) if and only if \( S(A) \cap S(B) \neq \emptyset \), that is, there is a vector \( c \in R^*_+ \) such that \( A^t c < e \) and \( B^t c < e \).

Applications of Theorem 3.4 depend crucially on our ability to show that two \( M \)-matrices have a common \( d \). To this effect we formulate a linear programming problem whose solution decides the existence of a suitable \( d \).

Theorem 3.6. Let \( A, B \in R^{n \times n} \) be two \( M \)-matrices, and \( e \in R^n \). Define the augmented matrices \( A, B \in R^{n \times (n+1)} \) such that \( A = [A, -e] \) and \( B = [B, -e] \). Let \( e \in R^n \) be defined as \( e = (1, 1, ..., 1)^T \), and the augmented vector \( x \in R^{n+1} \), such that \( z = (x_1, x_2, ..., x_{n+1})^T \). Consider the linear programming problem:
\[
\begin{align*}
\text{max} & \quad s_{n+1} \\
\text{s.t.} & \quad Ax \geq e, Bx \geq e, e \geq 0, x_{n+1}.
\end{align*}
\]
(3.4)

Then, \( S(A) \cap S(B) \neq \emptyset \) if and only if \( s_{n+1} = \max x_{n+1} = 1 \).

Proof. To prove sufficiency, let us assume that \( s_{n+1} = 1 \). Then, \( x \in R^*_+ \), such that \( Ax \geq e \) and \( Bx \geq e \). Furthermore, for sufficiently small \( z \geq 0 \) for which we have \(-zB < e\) and \(-zA < e\) we can set \( d = x + ze \). Then, \( Ad = Ax + ze \geq e + zAe \geq 0 \) and \( Bd = Bx + zBe \geq e + zBe > 0 \) so that \( d \in S(A) \cap S(B) \neq \emptyset \).

To establish necessity, let us assume that \( S(A) \cap S(B) \neq \emptyset \). Then, there exists a vector \( d \in R^*_+ \) such that \( Ad = c_1 \in R^*_+ \) and \( Bd = c_2 \in R^*_+ \). If \( c_1, c_2 > e \), then \( z = (d, e)^T \). It is obvious that this \( z \) is a solution of (3.4) with \( x_{n+1} = 1 \). If either \( c_1 \) or \( c_2 \) has elements less than one, then we find \( y = \max \{ y(x), y(c_2) \} \), where the function \( f : R^*_+ \rightarrow R_+ \) is defined as the smallest positive element of a positive vector. Then, the vector \( z = (d, y)^T \) is a solution of (3.4) with \( x_{n+1} = 1 \). \( \square \)

Remark 3.7. It is important to note that problem (3.4) is convex and, therefore, if there is a solution to (3.4), a linear programming method (e.g., simplex method) would find it.
Remark 3.3. The "dual" of Theorem (3.6) can be readily established by defining the augmented matrices $\tilde{A} = [A', -e]$ and $\tilde{B} = [B', -e]$. Thus, the linear programming problem is the same as (3.4) and we have $S(\tilde{A}) \cap S(\tilde{B}) \neq \emptyset$ if and only if $x''_{s+1} = 1$.

Another way to provide conditions for convexity of $M$-matrices is to use the Lapunov method based upon part (vi) of Theorem 2.1 and the idea proposed by Hornberger and Belanger [17] which are used in the following:

Definition 3.9. Let $A, B \in \mathbb{R}^{n \times n}$ be two $M$-matrices. We say that $A$ and $B$ have a common $D$ if there exists a matrix $D = \text{diag}(d_1, d_2, \ldots, d_n)$ such that both matrices

\[ C_1 = A - D \quad \text{and} \quad C_2 = B - D \]  

are positive definite.

The relevant result is the following:

Theorem 3.10. Let $A, B \in \mathbb{R}^{n \times n}$ be two $M$-matrices. If $A$ and $B$ have a common $D$, then the matrix $M(a)$ is an $M$-matrix for all $a \in [0, 1]$.

Proof. Observe that $M(a) \in L_{\mathbb{T}}$ for all $a \in [0, 1]$. Then,

\[ C(a) = M(a) - \text{diag}(aD + L(a)) - aL(a) + LA \]  

is positive definite for all $a \in [0, 1]$.

Remark 3.11. As in Theorem 3.2, Lemma 4, and Theorem 3.10, respectively, imply a more general property than announced. Namely, they guarantee that for any two numbers $\alpha, \beta \geq 0$, $\alpha + \beta > 0$, the matrix $M(\alpha, \beta) = \alpha M + \beta M$ is an $M$-matrix.

To determine if two $M$-matrices have a common $D$, we can solve a set of linear matrix inequalities [8,12]. We write $C \succ 0$ to indicate that the matrix $C$ is positive definite, and state the obvious:

Theorem 3.12. Two $M$-matrices $A, B \in \mathbb{R}^{n \times n}$ have a common $D$ if the following system of linear matrix inequalities:

\[ A' D = L \succ 0, \]

\[ B' D = L \succ 0, \]

\[ D \succ 0 \]  

is feasible.
Example 3.13. To demonstrate the fact that Theorems 3.4, 3.10 and Remark 3.5 present independent conditions for convexity of two $M$-matrices, we provide the following three pairs of matrices:

(i) $A = \begin{bmatrix} 0.6 & -0.4 \\ -1 & 0.8 \end{bmatrix}$ and $B = \begin{bmatrix} 0.6 & -0.4 \\ -0.1 & 0.1 \end{bmatrix}$ (3.9)

have a common $d$, but do not have either a common $\bar{d}$ or $D$

(ii) $A = \begin{bmatrix} 0.6 & -1 \\ -0.4 & 0.8 \end{bmatrix}$ and $B = \begin{bmatrix} 0.6 & -0.1 \\ -0.4 & 0.1 \end{bmatrix}$ (3.10)

have a common $\bar{d}$, but do not have either a common $d$ or $D$

(iii) $A = \begin{bmatrix} 0.4 & -0.1 \\ -0.7 & 0.5 \end{bmatrix}$ and $B = \begin{bmatrix} 0.4 & -0.6 \\ -0.4 & 0.7 \end{bmatrix}$ (3.11)

have a common $D$, but do not have either a common $d$ or $\bar{d}$.

Finally, we shall prove the most general result in this context announced by Horn and Johnson [16]:

Theorem 3.14. Let $A, B \in \mathbb{R}^{n \times n}$ be two $M$-matrices. Then, $M(a)$ is an $M$-matrix for all $a \in [0, 1]$ if and only if the matrix $B^{-1}A$ has no negative real eigenvalues.

Proof. Observe that $M(0) = B$ and $M(1) = A$ are $M$-matrices. If $a \in (0, 1)$, then we write

$M(a) = \begin{bmatrix} aB^{-1}A + (a^{-1} - 1)I \end{bmatrix}$. (3.12)

Now, since $M(0)$ and $M(1)$ are $M$-matrices and the eigenvalues of $M(a)$ depend continuously on $a$, we have that $M$-matrix property of $M(a)$ is equivalent to det $M(a) \neq 0$ for all $a \in (0, 1)$. This is the consequence of (viii) in Theorem 2.1 and the fact (e.g., [6]) that if $A \in \mathbb{M}_+$, then there exists a positive eigenvalue $\lambda(A)$ of $A$ such that the real part of any eigenvalue of $A$ is larger or equal to $\lambda(A)$. From (3.12), it follows that det $M(a) = 0$ is equivalent to det $B^{-1}A + (a^{-1} - 1)I = 0$. Since $a \in (0, 1)$, that is $(a^{-1} - 1) \in (0, +\infty)$, we conclude that det $M(a) \neq 0$ for all $a \in (0, 1)$ if and only if $B^{-1}A$ has no negative real eigenvalues. \qed

How to generalize Eq. (3.12) to convex polytopes of $M$-matrices, which is crucial in stability analysis of polytopic dynamic systems, is an open problem. For this reason, we shall use:

Corollary 3.15. Let $A, B \in \mathbb{R}^{n \times n}$ be two $M$-matrices. Then, $M(a)$ is an $M$-matrix for all $a \in [0, 1]$ if $B^{-1}A$ is an $M$-matrix.
Finally, we want to show that the convexity conditions for $M$-matrices carry over directly to the $H$-matrices.

**Theorem 3.16.** Let $A, B \in \mathbb{M}_n$ be two positive stable matrices. If $Q(A) + (1 - \alpha)Q(B)$ is an $M$-matrix for all $\alpha \in [0,1]$, then $\alpha A + (1 - \alpha)B$ is positive stable $H$-matrix for all $\alpha \in [0,1]$.

**Proof.** Since $A$ and $B$ are positive stable $H$-matrices, we have $a_{ij}, b_{ij} > 0$ for all $i \neq j$. Then,

$$|a_{ij} + (1 - \alpha)b_{ij}| = a_{ij} + (1 - \alpha)b_{ij} \quad \forall i \in n.$$  \hfill (3.13)

We also note that

$$|a_{ij} + (1 - \alpha)b_{ij}| \leq a_{ij}|a_{ij}| + (1 - \alpha)|b_{ij}| \quad \forall i, j \in n, i \neq j. \hfill (3.14)$$

or

$$|a_{ij} + (1 - \alpha)b_{ij}| \geq a_{ij}(-|a_{ij}|) + (1 - \alpha)\epsilon_{ij} \quad \forall i, j \in n, i \neq j. \hfill (3.15)$$

From (3.13) and (3.15) it follows that

$$Q(\alpha A + (1 - \alpha)B) \geq Q(A) + (1 - \alpha)Q(B) \in \mathbb{M}_n \quad \forall \alpha \in [0,1]. \hfill (3.16)$$

Since $Q(A) + (1 - \alpha)Q(B) \geq 0$ from (3.16) it follows that $Q(\alpha A + (1 - \alpha)B) \in \mathbb{M}_n$ for all $\alpha \in [0,1]$. Furthermore, the diagonal elements of the matrix $Q(\alpha A + (1 - \alpha)B)$ are positive, and we conclude that $\alpha A + (1 - \alpha)B$ is a positive stable $H$-matrix for all $\alpha \in [0,1]$. \hfill \Box

4. Polytopic linear systems

Let us consider a linear system

$$x = Ax, \hfill (4.1)$$

where $x(t) \in \mathbb{R}^n$ is the state of the system at time $t \in \mathbb{R}$, and $A \in \mathbb{R}^{n \times n}$ is a constant matrix. The uncertainty of the system is characterized by the fact that $A$ belongs to a polytope

$$A = \text{conv} \{ A_k \}, \hfill (4.2)$$

which is a convex hull of matrices $A_k \in \mathbb{R}^{n \times n}, k \in \{ 1, 2, \ldots, m \} = m$, called the generators of $A$. In terms of Liapunov's stability we state:

**Definition 4.1.** A polytopic linear system is stable if its equilibrium $x = 0$ is a globally asymptotically stable for all $A \in A$.

**Remark 4.2.** We recall that a linear system (4.1) is globally asymptotically stable if and only if the matrix $A$ has all eigenvalues with negative real parts, that is, it is a Hurwitz matrix. It is obvious that if $-A \in \mathbb{Z}_n$, then from (viii) of Theorem 2.1, $-A$ being Hurwitz is equivalent to $-A$ being an $M$-matrix. In the context of $M$-matrices it
is suitable to deal with positive stability, and we shall provide conditions under which $A \in M_k$, that is, the polytope $A$ is positive stable.

Let us note first that any matrix $A$ in $A$ can be expressed as a convex combination of the generators $A_0$ as

$$A(a) = \sum_{k=1}^{n} a_k A_k,$$

(4.3)

where the vector $a$ belongs to the unit simplex

$$U_m = \left\{ a \in \mathbb{R}^m_+ : \sum_{k=1}^{n} a_k = 1 \right\}.$$

(4.4)

Then we assume that $A \in Z_m$ and generalize Corollary 3.15 by proving:

**Theorem 4.3.** Let $A_k \in M_k$. Then, the matrix polytope $A$ is positive stable if $A_k^{-1} A_k \in \mathbb{M}_k$, for all $k < \ell$ and $k, \ell \in m$.

**Proof.** We prove the theorem by induction. Let $m = 2$. Then, $A(a) = a_1 A_1 + (1 - a_1) A_2$, $a_1 + a_2 = 1$, and by Corollary 3.15, $A_k^{-1} A_k \in \mathbb{M}_k$ implies that $A(a)$ is positive stable for all $a \in U_2$.

Now, we assume that the statement of the theorem holds for $m$ and then show that it also holds for $m + 1$. Let us rewrite (4.3) as

$$A(a) = a_1 A_1 + \sum_{k=2}^{n} a_k A_k.$$  

(4.5)

If $a_1 = 1$, we have $A(a) = A_1$, and $A(a)$ is positive stable. When $a_1 \neq 1$, we write

$$A(a) = a_1 A_1 + (1 - a_1) \sum_{k=2}^{n} \frac{a_k}{1 - a_1} A_k.$$  

(4.6)

Applying again Corollary 3.15 we conclude that $A(a)$ would be positive stable if

$$\sum_{k=2}^{n} \frac{a_k}{1 - a_1} A_k^{-1} A_k \in \mathbb{M}_n.$$  

(4.7)

We further note that

$$\sum_{k=2}^{n} \frac{a_k}{1 - a_1} = 1$$  

(4.8)

and the matrices

$$(A_k^{-1} A_k)^{-1} A_k^{-1} A_k = A_k^{-1} A_k$$  

(4.9)

are $M$-matrices. Then, by induction hypothesis (4.7) follows. This further implies that $A(a)$, as defined by (4.5), is an $M$-matrix for all $a \in U_m$, and the matrix polytope $A$ is positive stable. □
Let us provide an alternative proof of Theorem 4.3. From the assumption it follows that \( A_k^{-1} A_{k+1} \in M_k \) for \( k = 1, 2, \ldots, m - 1 \). Since \( A_m \in M_m \) there is a vector \( \bar{d} \in \mathbb{R}_+^m \) such that \( A_m^T \bar{d} = 0 \). Then,
\[
A_k^T = (A_k^{-1} A_{k+1})^T A_k^T \bar{d} > 0 \quad \forall k = 1, 2, \ldots, m - 1,
\]
(4.10)
because \( A_k^{-1} A_{k+1} \) is a nonnegative and non-singular matrix and \( A_k^T \bar{d} > 0 \). This implies that \( A_k^T \bar{d} > 0 \) for all \( k \in \mathbb{N} \) and thus,
\[
A_k^T (x_0 \bar{d}) > 0 \quad \forall x_0 \in U.
\]
(4.11)
Since \( A(x_0) \in Z_m \), (4.11) means that \( A \) is a polytype of M-matrices and, therefore, positive stable. From this alternative proof of Theorem 4.3, we have:

Theorem 4.4. Let \( A_k \in M_k \) for all \( k \in \mathbb{N} \). Then, the matrix polytype \( A \) is positive stable if there exists a generator \( A_k \) such that \( A_k^T A_{k+1} \in M_k \) for all \( k \in \mathbb{N} \), \( k \neq k' \).

The convexity conditions of Theorems 3.6, 3.12, and Remark 3.8, which were formulated in the mathematical programming context, generalize in an obvious way to the polytypes. From Theorem 3.6 we have:

Theorem 4.5. Let \( A_k \in M_k \) for all \( k \in \mathbb{N} \). Define the augmented matrices \( A_\infty \in \mathbb{R}^{N(m+1)} \) such that \( A_\infty = (A_k, -e) \) and the augmented vector \( \bar{x} \in \mathbb{R}^{m+1} \), such that \( \bar{x} = (x_1, x_2, \ldots, x_m, x_{m+1})^\top \). Consider the linear programming problem:
\[
\max \quad x_{m+1}
\text{s.t.} \quad A_\infty \bar{x} \geq 0, \quad \bar{x} > 0, \quad x_{m+1} \leq 1.
\]
(4.12)
Then, the matrix polytype \( A \) is positive stable if \( x_{m+1}^\ast = \max x_{m+1} = 1 \).

Similarly, from Theorem 3.12, we obtain:

Theorem 4.6. Let \( A_k \in M_k \) for all \( k \in \mathbb{N} \). Then, the matrix polytype \( A \) is positive stable if the following system of linear matrix inequalities
\[
A_k^T D_k + D_k A_{k+1} > 0, \quad k \in \mathbb{N},
\]
(4.13)
is feasible.

It is important to note that Theorems 4.3–4.6 which apply to the class \( Z_m \) and, thus, require nonnegativity of the off-diagonal elements of \( A_k \), generalize directly to the class \( H_m \) via Theorem 3.16. The removal of the sign restriction imposed by \( Z_m \) however, implies that the \( M \)-matrix conditions become only sufficient for \( A \) to be positive stable.

Let us broaden the scope of this section and provide an interesting interpretation of the obtained results in the context of systems with uncertain parameters, which are described as
\[
x = (A(p) x).
\]
(4.14)
The parameter vector \( p \in \mathbb{R}^n \) is assumed to belong to a polytope
\[
P = \text{conv} \{ p_i \},
\]
where \( p_i, i \in \mathbb{N} \), are the generators (vertex vectors) of \( P \). We also assume that system matrix \( A(p) \) has affine uncertainty structure, that is, each \( a_i(p) \) of \( A(p) \) is an affine function of \( p \),
\[
a_i(p) = \delta_i^0 p + \xi_i,
\]
with fixed vector \( \delta_i \in \mathbb{R}^n \) and scalar \( \xi_i \). Then, any \( p \in P \) can be expressed as a convex combination:
\[
p = \sum_{k=1}^{n} a_k p^k
\]
Set appropriate \( n \) in the unit simplex \( U_n \), and we write
\[
A(p) = A \left( \sum_{k=1}^{n} a_k p^k \right) = \sum_{k=1}^{n} a_k A(p^k).
\]

By recognizing that \( A_k = A(p^k) \) are the generators of the matrix polytope \( A \), we see that our results in this section have direct justifications in terms of the uncertainty polytope \( P \) in the parameter space. Once \( P \) is specified, we compose the generators \( A_k \), and determine positive stability of system (4.14) with respect to the polytope \( P \) by establishing stability of the corresponding matrix polytope \( A \) using any of the above theorems. This is the approach we take in the next section when we consider Lequeu-Volterra models with uncertain parameters.

Finally, we should show how the results of this section generalize the concept of connectivity stability [32,33]. In this concept, the coefficients \( a_i \) of the matrix \( A \) are expressed as
\[
a_i(E) = \begin{cases} 
\delta_i + \xi_i d_i & i = i_e \\
\xi_i d_{ij} & i \neq i_e
\end{cases}
\]
where \( \delta_i, \xi_i \in [0,1] \) are the uncertain elements of the (interval) interconnection matrix \( E \in \mathbb{R}^{n \times n} \). Roughly speaking, it was shown [32,33] that if the fixed matrix \( A = A(E) \) is an \( M \)-matrix, where \( E \) is the binary matrix representing the interconnection structure of \( A \), then the matrix \( A(E) \) is positive stable for all \( E \leq E_0 \). If we interpret \( e_{ij} \) as elements of the vector \( p \in \mathbb{R}^n \), then connectivity stability is a polytopic stability with \( P \) being the unit hypercube \( \delta p^0 \). Providing stability at the vertex \( E \) of the hypercube establishes stability throughout the entire hypercube \( P \). Using Liapunov's direct method, connectivity stability has been applied to a wide variety of nonlinear models in science and engineering [32,33]. A generalization of the concept to include interconnected systems with polytopic uncertainty set \( P \) is provided in Section 7.

We should note here that polytopic linear systems have been studied in general terms for quite some time; see the book by Barmish [3]. We restricted our attention to \( M \)-matrix polytopes because of the wealth of results available for this class of matrices, and because of matrices appear as standard representations in a wide variety of dynamic
models. For background information on robust stability and interval $M$-matrices see [1,27,50,31].

5. Polytopic Lotka–Volterra equations

Stability analysis of uncertain Lotka–Volterra population models have been considered in the context of consecutive stability and $M$-matrix theory to study changing strength of interconnections among species [33]. These models require special attention because the presence of uncertain parameters in the species interactions affects both stability and location of the equilibria. To capture this special feature of the models, the new concept of parametric stability has been introduced [18,19,33], which addresses the joint problem of equilibrium feasibility and stability under parameter uncertainty.

In this section, we shall consider models where the uncertainty set in the parameter space is a polytope, and the system matrix has affine uncertainty structure. Stability conditions will involve convex interval matrices and parametric Lyapunov functions. The parametrization is dictated by the fact that the location of the equilibria depends on the parameter values, and so does the corresponding Lyapunov function [18,19].

Let us consider the Lotka–Volterra model

$$\dot{x} = X(x + A(p)x),$$

where $x(t) \in \mathbb{R}^n$ is the species vector at time $t \in \mathbb{R}$, $p \in \mathbb{R}^m$ is the uncertain parameter vector, $X = \text{diag}(x_1, x_2, \ldots, x_n)$, $a \in \mathbb{R}^n$ is a constant vector, and $A : \mathbb{R}^m \to \mathbb{R}^{n \times n}$ is the community matrix. As in the preceding section, we assume that $p$ is located anywhere in the polytope $P$, which is defined in (4.15), and that $A(p)$ has full affine uncertainty structure (4.16). This implies that each $A(p)$ is a member of matrix polytope $\mathcal{A}$ defined in (4.2) as a convex hull of its generators $A_s = A(s^T p^T)$.

To establish the kind of global asymptotic stability of system (5.1), which was proposed in [18], we first recognize the fact that equilibria $\bar{x}(p)$ are solutions of the algebraic equation

$$A(p)x = -a$$

and are functions of the parameter vector $p$. Then, for a fixed $p \in P$, we denote by $x(t, x_0, p)$ the solution of the polytopic system (5.1) starting from $x_0$ at $t_0 = 0$, and state various requirements regarding the dynamics of the system in:

**Definition 5.1.** A polytopic Lotka–Volterra system is parametrically stable with respect to $\mathbb{R}^2$, if for all $p \in P$:

(i) There exists the unique equilibrium $\bar{x}(p) \in \mathbb{R}^n$.

(ii) $\mathcal{R}_p^E$ is an invariant region, that is, $x_0 \in \mathcal{R}_p^E$ implies $x(t, x_0, p) \subset \mathcal{R}_p^E$.

(iii) $\bar{x}(p)$ is stable in the sense of Lyapunov.

(iv) $\bar{x}(p)$ is globally attractive with respect to $\mathbb{R}^n$, that is, $x_0 \in \mathcal{R}_p^E$ implies $x(t, x_0, p) \to \bar{x}(p)$ as $t \to +\infty$.

As the following theorem shows, the properties of system (5.1) specified in Definition 5.1 all hinge on $-A(p)$ being an $M$-matrix:
Theorem 5.2. Let \(-A(p)\in \mathbb{Z}_p\) for all \(p\in P\). Then, a polytopic Lur'e-Volterra system is parametrically stable with respect to \(\mathbb{R}^p\) if \(-A(p)\in \mathbb{M}_p\) for all \(p\in P\).

Proof. If \(-A(p)\in \mathbb{M}_p\) for all \(p\in P\), then from (vii) of Theorem 2.1 we have that \(-A(p)\) exists and is nonnegative for all \(p\in P\), so that

\[ A(p) = A^T(p)A(p) \]  \hspace{1cm} (5.3)

is the unique positive equilibrium; this is (i) of Proposition 5.1. To establish parts (ii)-(iv), let us first define the matrix

\[ C(p) = A^T(p)\Delta(p) + \Delta(p)A(p) \]  \hspace{1cm} (5.4)

and note from (vii) of Theorem 2.1 that if for any fixed \(p\) the matrix \(-A(p)\) is an \(M\)-matrix, then there exists a positive diagonal Liapunov solution \(\Delta(p) = \text{diag}\{\delta_i(p)\}, \delta_i(p), \ldots, \delta_m(p)\}\) so that \(C(p)\) is negative definite. Now, we introduce a parametric Volterra-type Liapunov function \(v: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+\) as

\[ v(x,p) = \sum_{i=1}^{m} \delta_i(p)(x_i - \xi_i(p) - \xi_i(p)\ln[x_i/\xi_i(p)]) \]  \hspace{1cm} (5.5)

where we require that for each fixed \(p\in P\) and all \(i \in n\), \(\delta_i(p) > 0\). By taking the time derivative of \(v(x,p)\) with respect to Eq. (5.1), we obtain

\[ \dot{v}(x,p_{k+1}) = \sum_{i=1}^{m} \delta_i(p)(x_i - \xi_i(p))\dot{x}_i \]

\[ = [\mathbf{I} - \xi(p)]C(p)(x - \mathbf{A}(p)) \]  \hspace{1cm} (5.6)

Since \(-A(p)\) is an \(M\)-matrix for all \(p\in P\), as stated in the theorem, from (5.4) it follows that there exist positive numbers \(\delta_i(p)\) as elements of the diagonal matrix \(\Delta(p)\) such that \(C(p)\) is negative definite for each \(p\in P\). For fixed positive numbers \(\delta_i(p)\), the function \(v(x,p)\) is positive definite in \(\mathbb{R}^n\), that is \(v(x,p) > 0\) for all \(x \in \mathbb{R}^n\), \(p\in P\), and \(v(x,p)\to\infty\) when either \(x\to\infty\) or \(|x|\to\infty\). Therefore, the negative definiteness of \(C(p)\) implies parts (ii)-(iv) of Proposition 5.1. \(\Box\)

The crucial fact that makes Theorem 5.2 interesting is our ability to establish by a variety of finite tests presented in Theorems 4.3-4.6 that \(-A(p)\) is a positive stable (i.e., is an \(M\)-matrix) for all \(p\in P\). The tests involve the total of \(m^2\) generators \(A_i\) of the matrix polytope \(A\), which correspond to the vertices of the uncertain polytope \(P\). Parameter-dependent Liapunov functions for stability analysis of uncertain systems have been initiated by Barmish and DeMarco [4]. Recent extensions of the concept are presented in [5,12]. See also the book by Barmish [3].

6. Nonlinear matrix systems

In a wide variety of modeling situations [32,33], nonlinear matrix systems offer a general and, at the same time, flexible models of uncertain systems. These systems
are characterized by coefficients of the system matrix being functions of the state. It has been shown that Luenberger’s method is usable for stability analysis of such systems [33], allowing for a considerable sophistication in capturing system stability performance under uncertainty [36].

Our interest in this section is to broaden the concept of connective stability of nonlinear matrix systems [32] by permitting the uncertain parameters to vary within a convex polytope in the parameter space. We consider a nonlinear system

$$\dot{x} = A(x, p)x,$$  \hspace{1cm} (6.1)

where $x(t) \in \mathbb{R}^n$ is the state of the system at time $t \in \mathbb{R}$, $p \in \mathbb{R}^m$ is the uncertain parameter vector, and the matrix function $A : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^{n \times n}$ is smooth enough so that the solutions of (6.1) exist and are unique (for all initial conditions $(x_0, z_0) \in \mathbb{P} \times \mathbb{R}^n$ and $t \in \mathbb{R}$). We assume again that $p \in \mathbb{P}$, where $\mathbb{P}$ is the polytope defined in (4.15).

We further assume that $a_{ij}(x, p)$ and $[a_{ij}(x, p)]_{i \neq j}$ are convex functions of $p$, that is, for $x \in U_n$,

$$a_{ij}(x, p) = a_{ij} \left( x, \sum_{k=1}^{n} a_k p_k \right) \leq \sum_{k=1}^{n} a_k a_{ij}(x, p^k), \quad \forall i \in \mathbb{N},$$

$$[a_{ij}(x, p)]_{i \neq j} \leq \sum_{k=1}^{n} a_k [a_{ij}(x, p^k)], \quad \forall i, j \in \mathbb{N}, i \neq j, \forall x \in \mathbb{R}^n, \forall p \in \mathbb{P}. \hspace{1cm} (6.2)$$

In addition, we assume that there are matrices $\tilde{A}_i = (\tilde{a}_{ij})$ such that

$$a_{ij}(x, p^k) \leq \tilde{a}_{ij} < 0, \quad \forall i \in \mathbb{N}, \forall k \in \mathbb{M},$$

$$0 \leq [a_{ij}(x, p)]_{i \neq j} \leq \tilde{a}_{ij}, \quad \forall i, j \in \mathbb{N}, \forall k \in \mathbb{M}, \forall x \in \mathbb{R}^n. \hspace{1cm} (6.3)$$

We note that this last assumption means that $-\tilde{A}_i \in \mathbb{X}$ for all $k \in \mathbb{M}$ and, likewise, $-A(x) \in \mathbb{X}$, where

$$\tilde{A}(x) = \sum_{k=1}^{n} a_k \tilde{A}_k, \quad x \in U_n. \hspace{1cm} (6.4)$$

**Definition 6.1.** A polytopic nonlinear matrix system is parametrically stable if its equilibrium $x = 0$ is globally asymptotically stable for all $a \in \mathbb{P}$.

We prove the following polytopic generalization of a connective stability result [32, Theorem 1]:

**Theorem 6.2.** The equilibrium $x = 0$ of a polytopic nonlinear matrix system is parametrically stable if $-A(x) \in \mathbb{X}$, for all $a \in U_n$. 

Proof. Let us consider a parametric Liapunov function \( v(x, a) \) defined as:

\[
v(x, a) = \sum_{j=1}^{n} d_j(x) x_j,
\]

(6.5)

where we require that for each fixed \( a \in U_a \) all components \( d_j(x) \) of vector \( d(x) \) are positive numbers. Therefore, for each \( a \in U_a \), we can find \( v(x, a) \) as

\[
\phi_1(\|x\|, a) \leq v(x, a) \leq \phi_2(\|x\|, a), \quad \forall x \in \mathbb{R}^n.
\]

(6.6)

where

\[
\begin{align*}
\phi_1(\|x\|, a) &= d_{\min}(a) \|x\|, \\
\phi_2(\|x\|, a) &= \sum_{j=1}^{n} d_j(x) \|x\|, \\
d_{\min}(a) &= \min_j \{d_j(x)\}, \\
d_{\max}(a) &= \max_j \{d_j(x)\}
\end{align*}
\]

(6.7)

are the standard \( \mathcal{X} \)-functions of Hahn [14] for each fixed \( a \in U_a \). By using the functional defined by Rovenjreuk [28],

\[
\sigma_j = \begin{cases} 
1 & \text{if } x_i > 0, \text{ or if } x_i = 0 \text{ and } x_j > 0, \\
0 & \text{if } x_i = 0, \text{ and } x_j = 0, \\
-1 & \text{if } x_i < 0, \text{ or if } x_i = 0 \text{ and } x_j < 0
\end{cases}
\]

(6.8)

and taking the time derivative of \( v(x, a) \) with respect to (6.1), we get

\[
D^t v(x, a)(x, 1) = \sum_{i=1}^{n} d_i(x) \sigma_i x_i
\]

\[
= \sum_{i=1}^{n} d_i(x) x_i \sum_{j=1}^{n} a_j(x, p) x_j
\]

\[
= \sum_{i=1}^{n} d_i(x) x_i \sum_{j=1}^{n} a_j(x, p) x_j + \sum_{i=1}^{n} \sum_{j=1}^{n} d_i(x) a_j(x, p) x_j x_j
\]

\[
\leq \sum_{i=1}^{n} d_i(x) \sum_{k=1}^{n} a_k^2(x) x_k^2 + \sum_{i=1}^{n} \sum_{j=1}^{n} d_i(x) \sum_{k=1}^{n} a_k^2(x) x_k
\]

\[
= \sum_{i=1}^{n} d_i(x) \sum_{k=1}^{n} a_k^2(x) x_k^2
\]

\[
= \left( \sum_{k=1}^{n} a_k^2(x) \right) x(x)
\]

where

\[
x(x) = (x_1, x_2, \ldots, x_n)^T,
\]

\[\forall x \in \mathbb{R}^n, \forall a \in U_a\]

(5.9)
Now, we invoke the conditions of the theorem which, with part (v) of Theorem 2.1, guarantee that for each fixed \( x \in \mathcal{U}_0 \) there exists a positive vector \( d(x) \) such that
\[
d'(x)d(x) = -c^T(x),
\]
where \( c(x) \) is a positive vector as well. Then, from the last inequality in (6.9) we get
\[
D^xu(x, x, x, 1) \leq -\phi(x, x, x, x), \quad \forall x \in \mathbb{R}^n, \forall x \in \mathcal{U}_0,
\]
where
\[
\phi(x, x, x, x, x) = c_x(x)d(x), \quad c_x(x) = \min \{c_i(x)\}
\]
\( \Phi \) is the function of Hahn [14] for each \( x \in \mathcal{U}_0 \). Inequalities (6.6) and (6.12) imply that the equilibrium \( \dot{x} = 0 \) of the nonlinear matrix system is globally asymptotically stable for all \( x \in \mathcal{U}_0 \) that is, for all \( p \in P \) -- it is parametrically stable.

Again, as in the case of Theorem 5.2, the power of Theorem 6.2 lies in our ability to make the parametric Lyapunov function work for each \( p \) in the uncertainty polytope \( P \). The convexity assumption (6.2) effectively reduces the testing of the conditions
\[
-\dot{A}(x) \in M_xi_0 \text{ for all } x \in \mathcal{U}_0
\]
to testing of the M-matrix condition on a finite number of numerical matrices \( A_i \) -- the generators of the corresponding matrix polytope \( A \).

7. Polytopic connective stability

Originally, the connective stability concept [12] was introduced to study the effect of the size of interconnections on stability of large-scale systems composed of interconnected subsystems. The elements \( e_{ij} \) of the interconnection matrix \( E = (e_{ij}) \) represented the size of interconnections between pairs of subsystems rather than pairs of variables as discussed at the end of Section 4. What we want to do now is consider connective stability of large-scale systems and allow the interconnection parameters \( e_{ij} \) between the subsystems to reside in a convex polytope rather than a hypersphere.

Let us consider a system
\[
\dot{x} = f(x, x),
\]
where \( f(x) \in \mathbb{R}^n \) is the state at some \( t \in \mathcal{T} \), and the function \( f: \mathcal{T} \times \mathbb{R}^n \rightarrow \mathbb{R}^n \) is defined, bounded, and continuous on the domain \( \mathcal{T} \times \mathbb{R}^n \); so that solutions \( x(t; x_0) \) of (7.1) exist for all initial conditions \( (x_0, x_0) \in \mathcal{T} \times \mathbb{R}^n \) and \( t \in \mathcal{T}_0 \), where \( \mathcal{T} = \{t; \infty\} \), \( t \) is a number or a symbol \( -\infty \), and \( \mathcal{T}_0 = \mathcal{T}_0 + \infty \). We assume that
\[
f(t, 0) = 0 \quad \forall t \in \mathcal{T}
\]
and \( x = 0 \) is the unique equilibrium state of \( \mathcal{S} \).

Let us assume that \( \mathcal{S} \) is an interconnected system
\[
\dot{x}_i = g_i(x_i) + h_i(x_i), \quad i \in \mathbb{N}
\]
composed of \( N \) subsystems
\[
\dot{x}_i = g_i(x_i, x), \quad i \in \mathbb{N},
\]
where \( x(t) \in \mathbb{R}^n \) is the state of the subsystem \( \mathcal{S}_i \), the function \( g : T \times \mathbb{R}^n \rightarrow \mathbb{R}^n \) represents the dynamics of \( \mathcal{S}_i \), \( h : T \times \mathbb{R}^n \rightarrow \mathbb{R}^n \) is the interconnection of \( \mathcal{S}_i \), and the rest of the system \( \mathcal{S}_i \) and \( N = \{1, 2, \ldots, N\} \).

We make further assumptions about the subsystem stability and the size of interconnections. First, we consider a scalar function \( \eta : T \times \mathbb{R}^n \rightarrow \mathbb{R}_+ \) which is continuous and satisfies a Lipschitz condition in \( x_i \) that is, \( \delta_i > 0 \) such that
\[
|\eta(t, x_i) - \eta(t, x_i')| \leq \delta_i|x_i - x_i'|, \quad \forall t \in T, \forall x_i, x_i' \in \mathbb{R}^n \tag{7.5}
\]

Furthermore, the function satisfies the inequalities
\[
\phi(c, d) \leq \gamma(c, d), \quad \phi(c, d) \leq \gamma(c, d), \quad \forall (c, d) \in T \times R^n \tag{7.6}
\]
for some \( \phi, \psi : \mathbb{R} \rightarrow \mathbb{R}_+ \), \( \phi, \psi \in \mathcal{X} \), which means that the equilibrium \( x = 0 \) of each \( \mathcal{S}_i \) is globally asymptotically stable.

To determine the effect of interconnections on stability when the subsystem \( \mathcal{S}_i \) are connected to each other, we represent the interconnection functions as [33]
\[
h_i(t, x) = \sum_{j=1}^N c_{ij}h_j(t, x), \quad i \in N, \tag{7.7}
\]
where \( c_{ij} \in \mathbb{R}_+ \) are elements of the \( N \times N \) interconnection matrix \( E = (c_{ij}) \) representing the strength of coupling among the subsystems \( \mathcal{S}_i \). We further assume that there are nonnegative numbers \( \beta_j \) so that the individual interconnections \( h_j(t, x) \) bounded by
\[
||h_j(t, x)|| \leq \beta_j \phi_j(||x_j||), \quad \forall (t, x) \in T \times \mathbb{R}^n \tag{7.8}
\]
Now, we define an uncertainty polytope
\[
E = \text{conv}\{e_{ij}\}, \tag{9}
\]
which is a convex hull of vertex matrices \( E_k \), \( k \in m \), and want to formulate conditions for global asymptotic stability of the equilibrium \( x = 0 \) of \( \mathcal{S} \) for all \( E \in E \) in terms of the vertex matrices \( E_k \) — this is the polytopic connective stability.

Let us first define the \( N \times N \) test matrix \( W = (\nu_{ij}) \) as
\[
\nu_{ij} = \begin{cases} 1 - \epsilon_i \beta_j, & i = j, \\ -\epsilon_i \beta_j, & i \neq j \end{cases} \tag{7.10}
\]
implying \( W = W(E) \). Then, we compute the matrices \( \mathcal{W} = W(E_k) \), \( k \in m \), which form a polytope
\[
W = \text{conv}\{W_k\}, \tag{7.11}
\]
because the elements \( \nu_{ij} \) of \( W \) are affine functions of the elements \( \nu_{ij} \) of \( E \).

**Theorem 7.1.** The equilibrium \( x = 0 \) of the interconnected system \( \mathcal{S} \) is connectively stable for all \( E \in E \) if \( \mathcal{W} \in \mathcal{M}_E \) for all \( k \in m \), and \( W_k \in \mathcal{M}_L \) for all \( k < l \), and \( k, l \in m \).
Proof. We compute
\[
D^+\nu(t,x;\mathcal{E}) = \lim_{h \to 0^+} \frac{1}{h} [\nu(t+h, x + h|\mathcal{E}||x+h(t,x)) - \nu(t,x)]
\]
\[
\leq D^+\nu(t,x;\mathcal{E}) + \kappa|\mathcal{E}(t,x)|
\]
\[
\leq -\phi\alpha(|\mathcal{E}|) + \kappa \sum_{j=1}^{N} \epsilon_j \beta_j \phi_j(\|\mathcal{E}_j\|)
\]
\[
\leq -\sum_{j=1}^{N} \omega_j \phi_j(\|\mathcal{E}_j\|), \quad \forall (t,x) \in T \times \mathbb{R}^n. \tag{7.12}
\]

Let us consider a function
\[
\nu(t,x;\mathcal{E}) = d^2(\mathcal{E})(t,x)
\]
(7.13)
as a parameter-dependent Lyapunov function for system \( \mathcal{E} \), where \( d \in \mathbb{R}^n \) is a constant vector which depends on \( E \) and whose existence has yet to be established for each \( E \in E \). The function \( \nu: T \times \mathbb{R}^n \to \mathbb{R}^n \) is a vector Lyapunov function \( (20,22,24,33,35) \) \( \nu = (\nu_1, \nu_2, \ldots, \nu_N)^T \). Using (7.12) we get
\[
D^+\nu(t,x;\mathcal{E}) \leq -c^2(\mathcal{E}) \phi(\|\mathcal{E}\|), \quad \forall (t,x) \in T \times \mathbb{R}^n, \tag{7.14}
\]
where \( \phi = (\phi_1, \phi_2, \ldots, \phi_N)^T \), and
\[
c^2(\mathcal{E}) = -d^2(\mathcal{E})W(\mathcal{E}). \tag{7.15}
\]

Now, we want to show that for each \( E \in E \) the vector \( c^2(\mathcal{E}) \) is positive. We note that for any \( E \in E \) there exist real numbers \( a_k \geq 0 \), \( k \in \mathbb{R} \), such that
\[
E = \sum_{k=1}^{m} a_k E_k, \quad \sum_{k=1}^{m} a_k = 1. \tag{7.16}
\]
Since each coefficient \( w_j \) of \( W \) is an affine function of the coefficients \( a_j \) of \( E \), we have
\[
W(\mathcal{E}) = W \left( \sum_{k=1}^{m} a_k E_k \right) = \sum_{k=1}^{m} a_k W(E_k) = \sum_{k=1}^{m} a_k W_k \tag{7.17}
\]
and \( W \in W \). The conditions of the theorem imply via Theorem 4.3 that \( W \) is a \( Q \)-type of \( M \)-matrices. Thus, from (iii) of Theorem 2.1, we conclude that for each \( E \in E \) there is a vector \( d \in \mathbb{R}^N \) such that \( c \in R^N \). This further means that for each \( E \in E \) we have
\[
\phi(\|\mathcal{E}\|) \leq \nu(t,x;\mathcal{E}) \leq \phi(\|\mathcal{E}\|),
\]
\[
D^+\nu(t,x;\mathcal{E}) \leq -\phi(\|\mathcal{E}\|), \quad \forall (t,x) \in T \times \mathbb{R}^n, \quad \forall E \in E. \tag{7.18}
\]
where the $\phi$ functions are

$$\phi_{\epsilon}(\|x\|) = \sum_{j=1}^{N} d_j \phi_{\epsilon}(\|x_j\|),$$

$$\phi_{\epsilon}(\|z\|) = \sum_{j=1}^{N} c_j \phi_{\epsilon}(\|z_j\|),$$

$$\phi_{\epsilon}(\|\hat{z}\|) = \sum_{j=1}^{N} c_j \phi_{\epsilon}(\|\hat{z}_j\|).$$

(7.19)

The equilibrium $x = 0$ is globally asymptotically stable for all $\epsilon \in \mathbb{R}$, and the polytopic connective stability of $\mathcal{A}$ is established.

In formulating Theorem 7.1 we relied on Theorem 4.3. Similar theorems can be proved when Theorem 4.3 is replaced by Theorems 4.4, 4.5 or 4.6. We further note that the type of stability established in Theorem 7.1 can be extended to other types of dynamic systems and Lyapunov functions [20–22,24,25,33].

8. Conclusion

A number of sufficient conditions have been derived, which can be used to determine if a matrix polytope is contained inside the set of $M$-matrices by testing only the vertices of the polytope. The conditions are useful in establishing stability of systems with polytopic uncertainty structure, which appear as mathematical models of a wide variety of natural and man-made processes. Proofs of stability rely on parameter-dependent Lyapunov functions, which is a concept of recent interest. Future research should explore further the potential of convex $M$-matrices and parametric Lyapunov functions to study stability of stochastic and hereditary systems with parameter uncertainty.

References


