2. Diversity
Diversity In Wireless Radio

• Communication over a flat fading channel has poor performance due to significant probability that channel is in a deep fade.

• **Performance** is increased by providing more independent looks at the signal paths that fade independently.

• **Diversity** can be provided across time, frequency and space (Relate this to the three spreads that we talked about in the channel modeling section)

• Our goal here is how to exploit the added diversity in an efficient manner.
2: Diversity

**Baseline: AWGN Channel**

\[ y = x + w \]

Received Signal \quad Transmitted Symbol \quad Noise

**BPSK modulation:** \( x = \pm a \)

\[
pe = Q\left(\frac{a}{\sqrt{N_0/2}}\right) = Q\left(\sqrt{2 \cdot SNR}\right)
\]

SNR = \( \frac{a^2}{N_0} \)

Error probability decays **exponentially** with SNR.
2: Diversity

Gaussian Detection

\[ P\{y | x = u_A\} \quad P\{y | x = u_B\} \]

If \( y < (u_A + u_B)/2 \)
choose \( u_A \)

If \( y > (u_A + u_B)/2 \)
choose \( u_B \)
2: Diversity

**Rayleigh Flat Fading Channel**

\[ y = h \cdot x + w \]

\[ h : \text{CN} \ (0,1) \]

**BPSK:** \( x = \pm a \) Coherent detection.

Conditional on \( h \),

\[ Q \left( \sqrt{2 \cdot |h|^2 \cdot \text{SNR}} \right) \]

Averaged over \( h \),

\[ p_e = \frac{1}{2} \left( 1 - \sqrt{\frac{\text{SNR}}{1+\text{SNR}}} \right) \approx \frac{1}{4 \cdot \text{SNR}} \]

at high SNR.
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Rayleigh vs AWGN

![Graph showing the comparison between Rayleigh and AWGN channel models. The graph plots the bit error rate (Pe) against signal-to-noise ratio (SNR) in dB. The graph compares BPSK over AWGN, non-coherent orthogonal, and coherent BPSK.]
Typical Error Event

Conditional on $h$,

$$Q\left(\sqrt{2 \cdot |h|^2 \text{SNR}}\right)$$

When $|h|^2 \approx 1/\text{SNR}$ error probability is very small.
When $|h|^2 < 1/\text{SNR}$, error probability is large:

$$p_e \approx P\left(|h|^2 < \frac{1}{\text{SNR}}\right) \approx \frac{1}{\text{SNR}}$$

$$|h|^2 \sim \exp(1), \text{i.e.} \quad f_{|h|^2}(x) = e^{-x}$$

Typical error event is due to channel being in deep fade rather than noise being large.
BPSK, QPSK and 4-PAM

- BPSK uses only the I-phase. The Q-phase is wasted.
- QPSK delivers 2 bits per complex symbol.
- To deliver the same 2 bits, 4-PAM requires 4 dB more transmit power.
- QPSK exploits the available degrees of freedom in the channel better.

- A good communication scheme exploits all the available d.o.f. in the channel.
Time Diversity

- Time diversity can be obtained by *interleaving* and *coding* over symbols across different *coherent time* periods.

Coding alone is not sufficient!. Not effective over slow fading channel.
Example: GSM

- A frequency division duplex system with 25 MHz for the UL and 25 MHz for the DL
  - Bands are divided into 200 KHz sub-channel
  - Each sub channel is shared by 8 users in a time division fashion.
2: Diversity

Example: GSM ....

Voice is divided into frames of length 20 ms
- Each frame is compressed into 228 bits.
- A rate $\frac{1}{2}$ convolutional code (generator polynomials are $D^4+D^3+1$ and $D^4+D^3+D+1$) is used to encode the data → 456 bits.

To achieve time diversity, the coded bits in two such 20 ms speech frame are interleaved across 8 consecutive time slots assigned to a specific user (114 bits per slot)
- A delay of roughly 40 ms

Amount of time diversity limited by delay constraint and how fast channel varies.

To get full diversity of 8, needs $v > 30$ km/hr at $f_c = 900$Mhz.
2: Diversity
2: Diversity

Simplest Code: Repetition

After interleaving over $L$ coherence time periods,

$$y_l = h_l \cdot x_l + w_l \quad l = 1, K, L$$

**Repetition coding:** $x_l = x$ for all $l$

$$y = h \cdot x + w$$

$$y = [y_1, y_2, \ldots, y_L]^T, \quad h = [h_1, h_2, \ldots, h_L]^T, \quad w = [w_1, w_2, \ldots, w_L]^T$$

This is classic vector detection in white Gaussian noise.
For BPSK: $x = \pm a$

$u_A = +ah, \quad u_B = -ah$

$y = h^* \cdot y$

Is a sufficient statistic (match filtering).

Reduces to scalar detection problem:

$y = \|h\| \cdot x + \mathcal{W}$
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Deep Fades Become Rarer

\[
P \left( \| h \|^2 < \varepsilon \right) \approx \frac{1}{L!} \varepsilon^L
\]

\[
p_e \approx P \left( \| h \|^2 < \frac{1}{\text{SNR}} \right) 
\approx \frac{1}{L!} \frac{1}{\text{SNR}^L}
\]
2: Diversity

Performance

\[ p_e \]

\[ \text{SNR (dB)} \]

Wireless Communication Systems
Beyond Repetition Coding

• Repetition coding gets full diversity, but sends only one symbol every $L$ symbol times.
• Does not exploit fully the degrees of freedom in the channel. (analogy: PAM vs QAM)
• How to do better?
Example: Rotation code (L=2)

\(x_1, x_2\) are two BPSK symbols before rotation.

\[
\mathbb{P}\{x_A \rightarrow x_B|h_1, h_2\} = Q\left(\frac{||u_A - u_B||}{2 \sqrt{N_0/2}}\right) = Q\left(\sqrt{\frac{\text{SNR}/2 \cdot (|h_1|^2d_1^2 + |h_2|^2d_2^2)}{}}\right)
\]

where \(d_1\) and \(d_2\) are the distances between the codewords in the two directions.
Rotation vs Repetition Coding

Rotation code uses the degrees of freedom better!
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**Product Distance**

\[
P \{ x_A \rightarrow x_B | h_1, h_2 \} = Q \left( \frac{\text{SNR}}{2} \left[ |d_1|^2 |h_1|^2 + |d_2|^2 |h_2|^2 \right] \right)
\]

\[
P \{ x_A \rightarrow x_B \} \\approx P \left\{ |d_1|^2 |h_1|^2 < \frac{1}{\text{SNR}} \quad \& \quad |d_2|^2 |h_2|^2 < \frac{1}{\text{SNR}} \right\}
\]

\[
\approx \frac{1}{|d_1|^2 |d_2|^2} \text{SNR}^{-2}
\]

**Product distance** = \( |d_1| |d_2| \).

Choose the rotation angle to maximize the worst-case product distance to all the other codewords:

\[
\theta^* = \frac{1}{2} \tan^{-1} 2.
\]
Multiple antennas are used for transmission and or reception of the signal. Key parameter here is the separation between the antennas (coherence distance).
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**Receive Diversity**

![Receive Diversity Diagram]

\[ y = h \cdot x + w \]

- Use a number of receive antennas that are well separate (> **coherence distance**) to generate independent receptions of the transmitted signal.
- Same as repetition coding in time diversity, except that there is a further power gain.
- Optimal reception is via match filtering (also known **optimal beamforming**).
2: Diversity

Receive Diversity ....

- **Selection Diversity**: choose received signal with the largest received power, SNR, etc.
- **Switched Diversity**: choose an alternate receive antenna if the signal level falls below a certain threshold.
- **Linear Combining**: linearly combine a weighted copy of all received signals
- There is a dramatic improvement even with just two branches
2: Diversity

**Rx: Selection and Switched Diversity**

Selection Diversity

Switched Diversity

Wireless Communication Systems
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**Rx Diversity: Linear Combining**

\[
y_1 = h_1 \cdot a + n_1 \quad h_1 = A_1 \cdot e^{j\phi_1} \\
y_2 = h_2 \cdot a + n_2 \quad h_2 = A_2 \cdot e^{j\phi_2}
\]

**Equal Gain:** \[\alpha_1 = e^{-j\phi_1}, \quad \alpha_2 = e^{-j\phi_2}\]

**Maximal Ratio:** \[\alpha_1 = A_1 \cdot e^{-j\phi_1}, \quad \alpha_2 = A_2 \cdot e^{-j\phi_2}\]

**MMSE Training:** \[(\alpha_1, \alpha_2) = \arg \min_{\alpha_1, \alpha_2} |\alpha_1 y_1 + \alpha_2 y_2 - a|^2\]

**Decoding:** \[a = S (\alpha_1 y_1 + \alpha_2 y_2)\]
2: Diversity

Transmit Diversity

\[ y = h^* \cdot x + w \]

- Provide a diversity benefit to a receiver without having multiple receive antennas by using **multiple transmit antennas**.
- If the transmitter know the channel then transmit \( x = x \cdot \frac{h}{||h||} \). This will **maximizes the received SNR** by in-phase addition of signals at the receiver (transmit beamforming).
- Reduces to scalar channel: \( y = ||h|| \cdot x + w \), same as Rx beamforming.
- What happens if transmitter does not have explicit knowledge of the channel? Two kind of transmit diversity techniques:
  - Transmit diversity with feedback from the receiver
  - Transmit diversity without feedback from the receiver
    - Blind, i.e. no training
    - With feedforward information, i.e. with training.
- **Note:** transmitting the same symbol from all antennas does not work! Why?.
2: Diversity

Transmit Diversity with Feedback

- $w_1$ and $w_2$ are varied such that $|y(t)|^2$ is maximized.
- $w_1$ and $w_2$ are adapted based on feedback information from the receiver.
2: Diversity

Transmit Diversity with Frequency Weighting

- Use frequency weighting to mitigate the harm of scenario B.
- Simulate fast fading \(\rightarrow\) can use conventional channel coding and interleaving techniques.
- Suitable for slow fading or static channels.

\[
\theta(kT) = 2\pi f_m kT
\]
2: Diversity

**Tx Diversity with Antenna Hopping**

- At time $i$, $1 \leq i \leq N$, transmit $s$ from antenna $i$.
- Achieves a **diversity order** of $N$.
- **Bandwidth efficiency** is $1/N$. 

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Wireless Communication Systems
The channel code has a minimum *Hamming distance* $d_{\text{min}} \leq N$.

- At time $i$, transmit code symbol $c_i$ from antenna $i$.
- After receiving the $N$ code symbols, the receiver performance ML decoding to decode the received code code word.
2: Diversity

**Tx Diversity via Delay Diversity**

- Provides a diversity benefit by introducing intentional multipath (simulating a multipath channel).
- Receiver uses an **equalizer or MLSE** for detection.
- Provides diversity order of $N$, no loss in BW efficiency
- Provides very little diversity benefit, if the channel if multipath to begin with.

Wireless Communication Systems
Space-time Codes

- Space-time codes are designed specifically for the transmit diversity scenario.
- For each input symbol, the space-time encoder chooses the constellation points to simultaneously transmit from each antenna so that coding and diversity gains are maximized.
- **No cheating!** Total transmitted power is still the same as single transmit antennas case.
- Both flavors: trellis codes & block codes.
2: Diversity  

**Space-Time Block Codes: Alamouti Scheme**

- **Idea:**
  
  \[
  \begin{bmatrix}
  c_1 & c_2 \\
  \end{bmatrix}
  \rightarrow
  \begin{bmatrix}
  c_1 & -c_2^* \\
  c_2^* & c_1 \\
  \end{bmatrix}
  \]

- **Assumption:** channel is quasi-static.
2: Diversity

Decoding of STBC

- Received Signal:

\[ r_1 = h_1 c_1 + h_2 c_2 + n_1 \]

\[ r_2 = -h_1^* c_2 + h_2^* c_1 + n_2 \]

\[ \mathbf{r} = \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} = \begin{bmatrix} h_1 & h_2 \\ -h_1^* & -h_2^* \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} + \begin{bmatrix} n_1 \\ n_2 \end{bmatrix} = \mathbf{H} \cdot \mathbf{c} + \mathbf{n} \]

- \( \mathbf{H} \) is **orthogonal**: \( \mathbf{H}^* \mathbf{H} = \begin{bmatrix} |h_1|^2 & j \mathbf{h}_1^T h_2 \\ j \mathbf{h}_2^T h_1 & |h_2|^2 \end{bmatrix} \)

  - Projecting onto the two columns of the \( \mathbf{H} \) matrix yields:

\[ \mathbf{H}^* \mathbf{r} \]
Decoding of STBC ....

- The noise term $\tilde{n}$ is still **white** and $c_1$ and $c_2$ are detected **independently**.

- Double the symbol rate of repetition coding

- 3dB loss of received SNR compared to transmit beamforming.

- With $M$ receive antennas:

  $\tilde{r} = \sum_{i=1}^{M} H_i^* r_i \quad \|h_i\|^2 \epsilon \left\{ \right\}$

- With $M$ receive antennas, a diversity order of $2M$ is achieved.

- Only **simple linear processing** at the receiver is required.

- Complete **CSI** is required at the receiver. In practice **channel estimation** is used to obtain CSI.
Decoding of STBC ....

\[ \tilde{r} = \begin{bmatrix} \mathbf{H}_1^* & \mathbf{H}_2^* \end{bmatrix} \begin{bmatrix} \mathbf{r}_1^T \\ \mathbf{r}_2^T \end{bmatrix} \]

Wireless Communication Systems
A space-time code is a set of matrices \( \{X_i\} \).

Full diversity is achieved if all pairwise differences have full rank.

Coding gain determined by the determinants of \((X_i - X_j)(X_i - X_i)^*\).

Time-diversity codes have diagonal matrices and the determinant reduces to squared product distances.
2: Diversity

Frequency Diversity

\[ y[m] = \sum_{l=0}^{L-1} h_l x[m - l] + w[m] \]

- **Resolution of multipaths** provides diversity.
- Full diversity is achieved by sending one symbol every L symbol times.
  - Sounds like repetition coding 😞 → this is inefficient
- Sending symbols more frequently may result in intersymbol interference (ISI).
- Challenge is how to mitigate the ISI while extracting the inherent diversity in the frequency-selective channel.
2: Diversity

**Approaches**

- Time-domain equalization (eg. GSM)
- Direct-sequence spread spectrum (eg. IS-95 CDMA)
- Orthogonal frequency-division multiplexing OFDM (eg. IEEE 802.11a/g, Flash-OFDM, IEEE 802.16, IEEE 802.20)
ISI Equalization

• Suppose a sequence of uncoded symbols are transmitted. Can full diversity be achieved?
• Answer is **YES!**
• Maximum likelihood sequence detection (MLSD) is the optimal solution:
  – Performed using the Viterbi algorithm.
  – Complexity might be quite high
• Other suboptimal equalization techniques:
  – Linear equalizers: zero-forcing and MMSE
  – Decision feedback equalizers (DFE)
2: Diversity  **MLSD: Reduction to Transmit Diversity**

\[ y[1] = x[1] \cdot h_0 \]

\[ y[2] = x[2] \cdot h_0 + x[1] \cdot h_1 \]

\[ y[3] = x[3] \cdot h_0 + x[2] \cdot h_1 + x[3] \cdot h_2 \]

\[ y[4] = x[4] \cdot h_0 + x[3] \cdot h_1 + x[2] \cdot h_2 \]
Consider the transmitted sequence

\[ x^T = [x[1], x[2], \ldots, x[N + L - 1]] \]

\[ y^T = h^T \cdot X + w^T \]

\[ y^T = [y[1], y[2], \ldots, y[N + L - 1]] \]

\[ h^T = [h_0, h_1, \ldots, h_L], \quad w^T = [w[1], w[2], \ldots, w[N + L - 1]] \]

\[
X = \begin{bmatrix}
  0 & 0 & x[1] & x[2] & \cdots & \cdots & \cdots & \cdots \\
  \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
  0 & 0 & \cdots & \cdots & x[1] & x[2] & \cdots & x[N] \\
\end{bmatrix}
\]
MLSD: Reduction to Transmit Diversity …

- Maximum Likelihood detection of the sequence $x$ based on the received sequence $y$ (MLSD). The probability of confusing $x_A$ with $x_B$, when $x_A$ is transmitted is:

$$\Pr \{ x_A \rightarrow x_B \} = E \left\{ Q \left( \sqrt{\frac{\text{SNR} \cdot h (X_A - X_B) (X_A - X_B)^* h^*}{2}} \right) \right\}$$

$$\leq \prod_{i=1}^{L} \frac{1}{1 + \text{SNR} \cdot \lambda_i^2 / 4}$$

- The probability of error decays like SNR whenever the difference $X_A - X_B$ is of rank $L$ (i.e. full rank).
- Can show that is $X_A - X_B$ indeed of full rank (see details in text). Hence:

**MLSD achieves full diversity on symbol $x[N]$ using the observations up to time $N+L-1$, i.e. a delay of $L-1$ symbol times.**
• Data Rate $R$ is less than the transmission bandwidth $W$ (also known as chip rate).

**Processing Gain:** $n = \frac{W}{R}$

• Signal-to-noise ratio per chip is low.
• Example: IS-95 CDMA system (used by Verizon, Sprint):

  $W = 1.288 \text{ MHz}, \quad R = 9.6 \text{ kbits/sec}, \quad G = 128$
2: Diversity

Direct Sequence Spread Spectrum ....

Symbol duration $T_b$ >> delay spread $T_d$ → **ISI is not a problem** (as compared to interference from other users).

Channel is constant over one symbol. i.e. symbol duration $T_b = n/W <<$ coherence time $T_c$. 

Wireless Communication Systems
2: Diversity

**Direct Sequence Spread Spectrum** ....

- PN Sequences are chosen such that shifted versions of the same sequence are nearly orthogonal, *i.e.*:

\[
\langle (u'_l), (u'_k)^* \rangle = \langle (u'_l), (u'_l)^* \rangle = \sum_{i=1}^{n} |u[i]|^2 \quad l \neq k
\]

![PN Sequence Autocorrelation](image)
Direct Sequence Spread Spectrum ....

\[ y_i = \|u\| \cdot h_i \cdot s + w_i \quad l = 0, \ldots, L - 1 \]

RAKE Receiver

Wireless Communication Systems
2: Diversity

**Direct Sequence Spread Spectrum ....**

- Signal at the output of the correlators: \( y_l = \|u\| \cdot h_l \cdot s + w_l \) \( l = 0, \ldots, L - 1 \)
- The orthogonality \( u^l \) of implies that \( w_l \) are i.i.d \( XN(0,N_0) \).
- This looks exactly the same as the \( L \)-branch diversity model for the repetition code interleaved over time that we have seen before. Hence, the RAKE receiver in this case is nothing more a **ML ratio combiner** of the signals from the \( L \) multipath components (**\( L \) RAKE fingers**)
- Probability of error:
  \[
  p_e = E \left\{ Q \left( \sqrt{2 \|u\|^2 \sum_{l=0}^{L-1} |h_l|^2 / N_0} \right) \right\}
  \]
- Assume a Rayleigh fading model such that \( h_l \) are i.i.d \( XN(0,1/L) \), i.e. the energy is split among the \( L \) taps (and normalizing such that \( E\{|h_l|^2\}=1 \)). The SNR per branch is \( \text{SNR} = \|u\|^2/(N_0L) = (1/L) \cdot E_s/N_0 \)
- As \( L \to \infty \), \( |h_l|^2 \to 1 \) and the probability of error becomes:
  \[
  p_e \to Q \left( \sqrt{2E_s / N_0} \right)
  \]
  i.e. the **performance of AWGN with the same** \( E_s/N_0 \) **is asymptotically achieved**
ISI and Frequency Diversity

- In narrowband systems, ISI is mitigated using a complex receiver.

- In asynchronous CDMA uplink, ISI is there but small (compared to interference from other users).

- But ISI is not intrinsic in a channel with frequency diversity.

- The transmitter needs to do some work too!
OFDM: Basic Concept

- Most wireless channels are **under-spread** \((T_d << T_c)\).
- Can be approximated by a **linear time invariant** channel over a long time scale.
- Complex **sinusoids** are the only eigenfunctions of linear time-invariant channels.
- Should signal in the **frequency domain** and then transform to the time domain.
The channel Matrix $H$ is **circulant**, hence it will have the eigenvalue decomposition $H_k = Q^* \Lambda_k Q$.

FFT of received signal

$$Q \cdot y_k = Q \cdot Q^* \Lambda_k Q \cdot X_k + Q \cdot w$$

$$Y_k = \Lambda_k \cdot S_k + N_k$$
2: Diversity

**OFDM Modulation** ....

Data Symbols

\[ f_N = \frac{N}{T_s} \]
\[ f_2 = \frac{2}{T_s} \]
\[ f_1 = \frac{1}{T_s} \]

**Cyclic prefix**

\[ T'_s \]

**OFDM symbol**

\[ T_s = T'_s + T_v \]
2: Diversity

Tone Orthogonality

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2: Diversity

OFDM Modulation ....

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OFDM Modulation ....

OFDM transforms the communication problem into the frequency domain:

\[ y_i = h_i^0 s_i + w_i \quad i = 0, \ldots, N - 1 \]

a bunch of non-interfering (parallel) sub-channels, one for each sub-carrier.

\[ h_i^0 = H \left( \frac{i \cdot W}{N} \right) = H \left( i \cdot \Delta f \right) \quad i = 0, \ldots, N - 1 \]

Can apply time-diversity techniques.
Cyclic Prefix Overhead

• OFDM overhead
  \[ \text{OFDM overhead} = \frac{\text{length of cyclic prefix}}{\text{OFDM symbol time}} \]
• Cyclic prefix dictated by delay spread.
• OFDM symbol time limited by channel coherence time.
• Equivalently, the inter-carrier spacing should be much larger than the Doppler spread.
• Since most channels are underspread, the overhead can be made small.
Diversity Summary

- Fading makes wireless channels unreliable.
- Diversity increases reliability and makes the channel more consistent.
- Smart codes yields a coding gain in addition to the diversity gain.
- Different diversity schemes for different channel conditions:
  - **Time Diversity**: exploit the diversity inherent in the time varying nature of the channel
  - **Frequency Diversity**: exploit the multipath (frequency) in the channel
  - **Space Diversity**: exploits the spatial variation in the channel
2: Diversity

The Big Picture So Far

• In wireless communications, information are sent over wireless channels
• Impairments of Wireless channels:

  – **Path Loss and Shadowing:**

    *Link budget and cell planning*

  – **Multipath Fast Fading:**

    *Use diversity (time, frequency, space) techniques.*